Hierarchical Dependability Models based on Non-Homogeneous Continuous Time Markov Chains

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***Abstract*—This paper shows a method of calculating the hazard rate of the non-homogeneous Markov chains using different homogeneous probability matrices for several hundreds small time intervals. The proposed method is applied on hierarchical dependability models allowing independent calculations of the hazard rates of multiple cooperating blocks of the system. The independent calculations are significantly faster than the calculation of a single model composed of all models of the blocks and the proposed method is very accurate compared to methods based on homogeneous Markov chains.**

***Index Terms*—Fault tolerant systems, Hierarchical systems, Reliability.**

I. Introduction

In this paper, we want to introduce a method capable to calculate hazard (failure) rate of a system modeled by a non-homogeneous Markov chain. Such method allows us to build hierarchical dependability models without simplifications and inaccuracies meant in [1].

The proposed method is demonstrated on a case study containing multiple (up to 9) identical dependable blocks configured as an N-modular redundant system (NMR). Models of the internal block redundancy used in the study systems are used as dependability models of railway/subway interlocking equipment used in Czech Republic. The case study is used to calculate the total hazard rate of the system and to demonstrate the dependencies of time-consumption and accuracy of the proposed method.

The paper is organized as follows: Section II introduces Markov chains. Section III describes the proposed method. The results are shown in Section IV and Section V concludes the paper.

II. Markov Chains Types and Conversions

There are two main types of the Markov chains: **Continuous Time Markov Chain** and **Discrete Time Markov Chain**. They are defined as the transition rate matrix Q and probability matrix P respectively, where the elements define the rate (probability) of transition from state i to state j.

These two types of MCs can be mutually converted. We use discrete time MC to continuous time MC conversion in our method that is based on the following equation:

$$p\left(\frac{1}{delta}\right) = \frac{q}{delta} \tag{1}$$

where $delta$ is a parameter determining how small the time interval will be and $p(\frac{1}{delta})$ is the probability of the event during the interval $(0, \frac{1}{delta})$ measured in hours. The smaller the interval, the more accurate is the conversion.

Our method requires the evolution of the state probabilities over time to be calculated. This evolution can be described by the Chapman-Kolmogorov equation [2]:

$$P(t + \Delta t) = P(t)P(\Delta t) \tag{2}$$

Most methods used to calculate the state probabilities and their evolution of over time can be used in homogeneous case only, but there are methods able to estimate the non-homogeneous MC by a homogeneous one. Two estimations are introduced in [3]: *Reduction to the homogeneous process* and *Constant rate matrix between different times*.

Both methods will lead to inaccurate solutions, but we use the second method modified to use several hundreds intervals, thus a very small error can be achieved. Each interval (time-slice) has a constant probability/transition matrix, ie. the MC is homogeneous in this time-slice.

III. Proposed method description

The proposed method has two main parts:

1) Calculate the failure distribution function of non-homogeneous model.
   In our case, we will calculate several hundreds of discrete samples of this function using Chapman-Kolmogorov equation (2) shown in Section II.
2) Calculate the hazard rate for each time-slice (interval between two consequential samples) calculated in the previous part.

A. Calculation of the failure distribution function of non-homogeneous model

The flowchart of the method is shown in Figure 1.

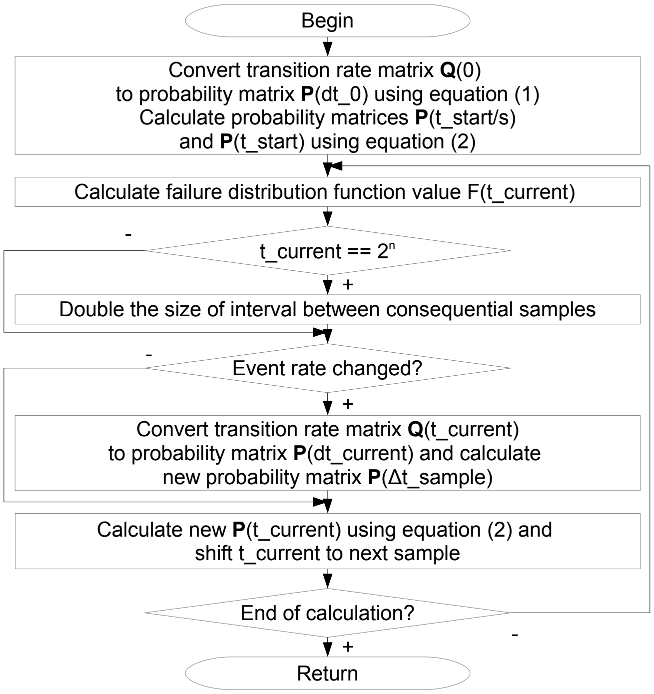

Fig. 1. Flowchart of calculation of failure distribution function of non-homogeneous model.

B. Calculate the hazard rate for each time-slice

We assume that the hazard rate is constant in the duration of the time-slice (ie. between two consequential samples of $F(t)$). This assumption will introduce an inaccuracy to our solution, but we can use as many samples as we need to obtain a suitable solution.

Now we take two samples bounding the i-th interval calculated from the previous part stored as pairs $\{t_i, F(t_i)\}$ and $\{t_{i+1}, F(t_{i+1})\}$ and simply create a system of two equations:

$$F(t_i) = 1 - e^{(-\lambda_i \,.\, t_i + \mathit{offset})} \tag{3}$$

$$F(t_{i+1}) = 1 - e^{(-\lambda_i \,.\, t_{i+1} + \mathit{offset})} \tag{4}$$

where *offset* is a constant that will be eliminated in the next step.

Assuming $t_i \neq t_{i+1}$ and $F(t_i) < F(t_{i+1}) < 1$, there is only one solution of this system:

$$\lambda_i = \frac{\log_e(1 - F(t_i)) - \log_e(1 - F(t_{i+1}))}{t_{i+1} - t_i} \tag{5}$$

IV. Results

The proposed method is demonstrated on a case study containing multiple (up to 9) identical dependable blocks (Two-out-of-two – 2oo2) configured as an N-modular redundant system (NMR). Models of the internal block redundancy used in the study systems are used as dependability models of railway/subway interlocking equipment used in Czech Republic [4].

The model of a 2oo2 block is created, the samples of the $F(t)$ function are calculated, and the hazard (failure) rate result (λ) is taken as the hazard rate of the NMR model.

Table I shows the comparison of the CPU-times and the relative errors of the hierarchical and the exact (the model generated by the Cartesian product of the dependability models of the 2oo2 blocks and the model of the NMR) solutions. The first column shows the number of the 2oo2 blocks, the second column shows the CPU-time[1] spent on exact model solution. The CPU-time spent on hierarchical method (CPU-times spent on the 2oo2 model and the NMR model) is shown in the third column. The relative error of the first sample (compared to the sample taken from the exact case) is shown in the third column. The average of the absolute values of relative errors of all stored samples is shown in the last column.

TABLE I
Comparison of CPU-times of exact and hierarchical case.

NMR blocks	Exact time [s]	Hierarchical time [s]	Rel. error of the first sample [-]	Average rel. error [-]
n03	1.160	0.091	1.77×10^{-7}	2.94×10^{-8}
n05	37.23	0.130	5.31×10^{-7}	8.67×10^{-8}
n07	627.5	0.169	1.06×10^{-6}	1.73×10^{-7}
n09	5,938	0.225	1.77×10^{-6}	2.85×10^{-7}

V. Conclusions

This paper presents a method able to calculate the samples of failure distribution function $F(t)$ from a non-homogeneous Markov chain used to calculate a hazard (failure) rate λ of a hierarchical Markov chain. The method allows accurate calculations even in the case of large complex systems, where the results of classical models are practically unreachable due to state explosion.

The method is significantly faster than a common non-hierarchical approach based on a complex Cartesian-product homogeneous model (up to ca. 10,000 times in the presented case studies based on dependable blocks configured as N-modular redundant system) and very accurate (the average relative error between proposed and common method is ca. 10^{-6} in the presented studies).

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