Theoretical Limitations in Vehicle System Dynamics

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Abstract: - It is often assumed that if practical difficulties are neglected, active systems could in principle produce arbitrary ideal behavior. This paper presents the factorization approach that is taken to derive limitations of achievable frequency responses for active vehicle suspension systems in terms of invariant frequency points and restricted rate of decay at high frequencies. The factorization approach enables to determine complete sets of such constraints on various transfer functions from the load and road disturbance inputs for typical choices of measured outputs and then choose the "most advantageous" vector of the measurements from the point of view of the widest class of the achievable frequency responses. Using a simple linear two degree-of-freedom car suspension system model it will be shown that even using complete state feedback and in the case of in which the system is controllable in the control theory sense, there still are limitations to suspension performance in the fully active state. In control law design for active suspension system of vehicles it is demanded to prevent magnitudes of the road and load frequency responses from being too large. The paper aims to show that there are some frequency points and frequency ranges where the transfer functions have modulus strictly greater than one i.e. where road and load disturbance amplification occurs. On the base of the Bode integral theorems a proof will be given to show that the transfer functions must be greater in modulus to at least the same extend that it is less than one, when measured in terms of the area on a Bode magnitude plot. In such a case there is a possibility to shift frequency ranges where disturbance amplification occurs to a "more advantageous place" or to make magnitudes lower spreading the frequency range.

Key-words: - one-quarter-car model, vibration, active suspension, comprime factorization, frequency responses invariant points, dynamics

1 Introduction

Two major performance requirements of suspension are to improve ride and handling quality when random road and load disturbances from the environment act upon running vehicles. Automotive suspensions are designed to provide good vibration insulation of the passengers and to maintain adequate adherence of the wheel for braking, accelerating and handling, i.e. the purpose of active suspensions in terms of performance is to improve both of these conflicting requirements.

In this paper it will be shown the factorization approach taken to derive limitations of achievable frequency responses for active vehicle suspension systems. As we will see, limitations derived for a traditional one-quarter-car model in the frequency domain arise in the form of invariant frequency points and restricted rate of decay at zero and infinite frequencies.

Youla-Kucera factorization approach to feedback system stability has been shown in [2)], [3] to derive achievable dynamic responses for active suspension systems of vehicles. Complete sets of constraints on various transfer functions from the road and load disturbance inputs were derived for typical choices of measured outputs.

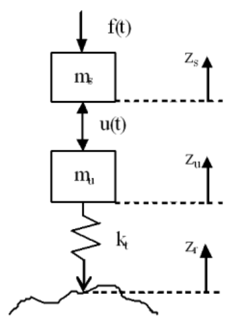

Fig.1.: One-quarter-car model

The approach was illustrated for the one-quarter-car model shown in Fig.1, where:

- $u(t)$ control input (active suspension force) [N]
- m_u weight of the unsprung mass (wheel) [kg]
- m_s weight of the sprung mass supported by each wheel and taken as equal to a quarter of the total body mass [kg]
- k_t stiffness of the tyre [N/m]
- $z_r(t)$ road displacement (road disturbance) [m]
- $z_s(t)$ displacement of the sprung mass [m]
- $z_u(t)$ displacement of the unsprung mass [m]
- $f(t)$ load disturbance [N]

Note that if the one-quarter-car model contains also a passive suspension system (a sprung of stiffness *k* and a shock absorber of damping quotient *b* then the suspension force *u(t)* involves also the adequate force generated by the passive suspension system.

On the base of the Youla-Kucera parametrization, complete sets of limitations were derived for transfer functions from the road disturbance input:

$$H^{1}_{zw}(s) = \frac{Z_s(s)}{Z_r(s)} /_{f=0} \quad \text{(to the sprung mass position)} \qquad (1)$$

$$H^{2}_{zw}(s) = \frac{Z_s(s) - Z_u(s)}{Z_r(s)} /_{f=0} \quad \text{(to the suspension deflection)} \qquad (2)$$

$$H^{3}_{zw}(s) = \frac{Z_u(s) - Z_r(s)}{Z_r(s)} /_{f=0} \quad \text{(to the tyre deflection)} \qquad (3)$$

and analogically for the load disturbance input for various choices of measured outputs even for full state feedback.

2 Comprime Factorization

Consider the standard feedback configuration shown in Fig.2,

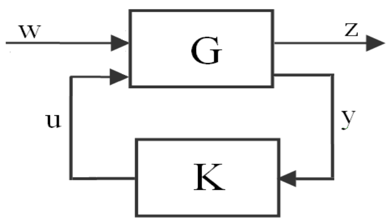

Fig. 2.: Standard feedback configuration

where *w* is the exogenous input, typically consisting of disturbances and sensor noises, *u* is the control signal, *z* is the output to be controlled, and *y* the measured output. In general *u*, *w*, *y* and *z* are vector-valued signals.

The transfer matrices *G(s)* and *K(s)* are, by assumption, real-rational and proper: *G* represents a generalized plant, the fixed part of the system, and *K* a controller [4]. Partition *G(s)* as:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \qquad (4)$$

Then Fig.2 stands for the following algebraic equations:

$$Z(s) = G_{11}(s)W(s) + G_{12}(s)U(s) \qquad (5)$$

$$Y(s) = G_{21}(s)W(s) + G_{22}(s)U(s) \qquad (6)$$

$$U(s) = K(s)Y(s) \qquad (7)$$

Manipulating the equations listed above, the following transfer function $T_{zw}(s)$ from *w* to *z* as a linear-fractional transformation of *K(s)* can be derived:

$$\begin{aligned} T_{zw} &= G_{11} + G_{12}K\left[I - G_{22}K\right]^{-1}G_{21} = \\ &= G_{11} + G_{12}\left[I - KG_{22}\right]^{-1}KG_{21} \end{aligned} \qquad (8)$$

It is shown in [1] that the set of all proper real-rational matrices *K(s)* stabilizing *G(s)* is parametrized by a free parameter $Q(s) \in RH_\infty$ as follows:

$$\begin{aligned} K &= \left[Y - MQ\right]\left[X - NQ\right]^{-1} = \\ &= \left[\tilde{X} - Q\tilde{N}\right]^{-1}\left[\tilde{Y} - Q\tilde{M}\right] \end{aligned} \qquad (9)$$

where $M(s), N(s), X(s), Y(s) \in RH_\infty$, $\tilde{M}(s), \tilde{N}(s), \tilde{X}(s), \tilde{Y}(s) \in RH_\infty$ can be found by comprime factorization approach of $G_{22}(s)$:

$$G_{22}(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s) \qquad (10)$$

$$\begin{bmatrix} \tilde{X}(s) & -\tilde{Y}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = I \qquad (11)$$

Substituting the equation (9) into (8) we obtain the transfer matrix $T_{zw}(s)$ from *w* to *z* in terms of the free parameter $Q(s) \in RH_\infty$:

$$\begin{aligned} T_{zw}(s) &= \\ &= G_{11}(s) + G_{12}(s)M(s)\left[\tilde{Y}(s) - Q(s)\tilde{M}(s)\right]G_{21}(s) = \\ &= G_{11}(s) + G_{12}(s)\left[Y(s) - M(s)Q(s)\right]\tilde{M}(s)G_{21}(s) \end{aligned} \qquad (12)$$

As the parameter *Q(s)* varies over the set of all stable proper functions, the equation (12) parametrizes all achievable transfer functions $T_{zw}(s)$.

If it is assumed that the tyre does not leave the ground, for the one-quarter car model (Fig.1) the linear differential equations of motion are:

$$m_s\ddot{z}_s = u - f \qquad (13)$$

$$m_u\ddot{z}_u = -u + k_t(z_r - z_u) \qquad (14)$$

where z_u and z_s are measured from the static equilibrium position.

First, let the load disturbance is absent (*f*=0). Adding equations (13) and (14) we obtain the invariant equation of:

$$m_s\ddot{z}_s + m_u\ddot{z}_u = k_t(z_r - z_u) \qquad (15)$$

that is independent on the suspension force *u*. The following transfer functions will be investigated:

$$H_{SP}(s) = Z_s(s)/Z_r(s) \qquad (16)$$

$$H_{SD}(s) = \left[Z_s(s) - Z_u(s)\right]/Z_r(s) \qquad (17)$$

$$H_{TD}(s) = \left[Z_u(s) - Z_r(s)\right]/Z_r(s) \qquad (18)$$

3 Invariant Properties

Manipulating the equation (15) we can derive the following invariant identities:

$$\left[(m_u + m_s)s^2 + k_t\right]H_{SP}(s) - \left[m_u s^2 + k_t\right]H_{SD}(s) = k_t \qquad (19)$$

$$m_s s^2 H_{SP}(s) + \left[m_u s^2 + k_t\right]H_{TD}(s) = -m_u s^2 \qquad (20)$$

$$\left[(m_u + m_s)s^2 + k_t\right]H_{TD}(s) + m_s s^2 H_{SD}(s) = = -(m_u + m_s)s^2 \quad (21)$$

It is obvious from (19) and (20) that the sprung mass position transfer function H_{SP} has an invariant "tyre-hop" frequency at $\omega_1 = \sqrt{k_t / m_u}$, where:

$$H_{SP}(s)/_{s=j\omega_1} = -m_u / m_s \quad (22)$$

Similarly from (16) and (18) the suspension deflection transfer function H_{SD} has an invariant "rattle-space" frequency at $\omega_2 = \sqrt{k_t /(m_u + m_s)}$ and:

$$H_{SD}(s)/_{s=j\omega_2} = -(1 + m_u / m_s) \quad (23)$$

Finally, from (20) and (21), the tyre deflection transfer function H_{TD} does not have any invariant frequency point except $\omega_3 = 0$, where:

$$H_{TD}(s)/_{s=j\omega_3} = 0 \quad (24)$$

4 Transfer Function Limitations

In the next, we will consider the standard block diagram shown in Fig.2. As an example, let $w=z_r$, $z=z_s$ and $\underline{y} = \left[z_u, z_s - z_u, z_u - z_r\right]^T$.

Then:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = = \begin{bmatrix} 0 & \dfrac{1}{m_s s^2} \\ \begin{bmatrix} \dfrac{sk_t}{m_u s^2 + k_t} \\ \dfrac{-k_t}{m_u s^2 + k_t} \\ \dfrac{-m_u s^2}{m_u s^2 + k_t} \end{bmatrix} & \begin{bmatrix} \dfrac{-s}{m_u s^2 + k_t} \\ \dfrac{(m_u + m_s)s^2 + k_t}{m_s s^2 (m_u s^2 + k_t)} \\ \dfrac{-1}{m_u s^2 + k_t} \end{bmatrix} \end{bmatrix} \quad (25)$$

The limitations of all achievable closed-loop transfer functions $T_{zw}(s) = H_{SP}(s)$ are derived from the right and left comprime factorization of $G_{22}(s)$, i.e.:

$$G_{22} = \underbrace{\begin{bmatrix} \dfrac{-m_s s^3}{p_4(s)} \\ \dfrac{(m_u + m_s)s^2 + k_t}{p_4(s)} \\ \dfrac{-m_s s^2}{p_4(s)} \end{bmatrix}}_{N(s)} \underbrace{\left[\dfrac{m_s s^2 (m_u s^2 + k_t)}{p_4(s)}\right]^{-1}}_{M^{-1}(s)} = = \underbrace{\begin{bmatrix} \dfrac{m_s s}{p_2(s)} & \dfrac{m_s s^2}{p_2(s)} & 0 \\ 0 & \dfrac{m_u m_s s^2}{p_2(s)} & \dfrac{-m_s k_t}{p_2(s)} \\ \dfrac{-1}{p_2(s)} & 0 & \dfrac{s}{p_2(s)} \end{bmatrix}^{-1}}_{\tilde{M}^{-1}} \underbrace{\begin{bmatrix} \dfrac{1}{p_2(s)} \\ \dfrac{m_u + m_s}{p_2(s)} \\ 0 \end{bmatrix}}_{\tilde{N}(s)} \quad (26)$$

where $p_2(s)$ and $p_4(s)$ are Hurwitz polynomials of degree 2 and 4, respectively. Then:

$$H_{SP}(s) = = G_{11}(s) + G_{12}(s)M(s)\left[\tilde{Y}(s) - Q(s)\tilde{M}(s)\right]G_{21}(s) = = -\frac{1}{p_4(s)}\tilde{Y}(s)\begin{bmatrix} -sk_t \\ k_t \\ m_u s^2 \end{bmatrix} + \frac{s(m_u s^2 + k_t)}{p_4(s)}Q^*(s) \quad (27)$$

where $Q^*(s) = \left[Q_3(s)/p_2(s)\right]$, $Q(s) = \left[Q_1(s) \quad Q_2(s) \quad Q_3(s)\right]$.

It follows from (27) that thanks to the term $\left[\tilde{Y}(s) - Q(s)\tilde{M}(s)\right]$ bounded for large s:

$$\lim_{s\to\infty} s^2 H_{SP}(s) < \infty \quad (28)$$

i.e., the resulting rate of decay is of second degree:

$$H_{SP}(s)/_{s\to\infty} = O(s^{-2}) \quad (29)$$

It is obvious from (27) that the member $\dfrac{s(m_u s^2 + k_t)}{p_4(s)}$ has two imaginary axis zeros - at $s=0$ and $s = j\omega_1 = j\sqrt{k_t / m_u}$ - which can not be canceled by the denominator of $Q^*(s) \in RH_\infty$. With respect to (27) and the Bezout identity (11) it follows, that:

$$H_{SP}(s)/_{s=0} = -\frac{1}{p_4(s)}\tilde{Y}(s)\begin{bmatrix} -sk_t \\ k_t \\ m_u s^2 \end{bmatrix}/_{s=0} = = -\frac{k_t}{p_4(s)}\tilde{Y}_2(s)/_{s=0} = 1 \quad (30)$$

and similarly:

$$H_{SP}(s)/_{s=j\omega_1} = -\frac{1}{p_4(s)}\tilde{Y}(s)\begin{bmatrix} -sk_t \\ k_t \\ m_u s^2 \end{bmatrix}/_{s=j\omega_1} = = -m_u / m_s \quad (31)$$

This result endorses (25).

Thanks to the first order of the imaginary axis zero at $s=0$, the first derivative of the transfer function $H_{SP}(s)$ does not have any similar restrictions at this point. Expressions (29), (30) and (31) create the complete set of limitations which any admissible transfer function $H_{SP}(s) \in RH_\infty$ must satisfy. Another words, if any complex transfer function satisfies the mentioned limitations, there exists a stabilizing controller $K(s)$ so that $T_{zw}(s) = H_{SP}(s)$. It does not depend on what variables are chosen as the measured output - the limitations always arise in the form of invariant frequency points as was shown in paragraph 3 and in the form of restricted rate of decay at infinite frequencies as shown in (29).

Complete sets of limitations for the transfer functions $H_{SD}(s)$ and $H_{TD}(s)$ can be similarly carried out from the corresponding transfer functions $G(s)$ or using (26), (27), (28) and the corresponding invariant equation stated above. That way the following complete sets of limitations can be derived:

$$H_{SD}(s)/_{s\to\infty} = O(s^{-2}) \quad (32)$$

$$H_{SD}(s)/_{s=0} = 0, \quad H_{SD}(s)/_{s\to 0} = O(s) \quad (33)$$

$$H_{SD}(s)/_{s=j\omega_2} = -(1+m_u/m_s)$$
$$\omega_2 = \sqrt{k_t/(m_u+m_s)} \quad (34)$$

$$H_{TD}(s)/_{s\to\infty} = -1+O(s^{-2}) \quad (35)$$

$$H_{TD}(s)/_{s=0} = 0,$$
$$H_{TD}(s)/_{s\to 0} = -(m_u+m_s)s^2/k_t + O(s^3) \quad (36)$$

Note, that even though it is desirable to prevent amplitudes of the frequency responses $H_{SP}(s)$, $H_{SD}(s)$, and $H_{TD}(s)$ being too large in any frequency domain, a brief analysis of the expressions (29) - (36) enables to find out that the investigated transfer functions must have modulus strictly grater that one at some frequencies what indicates the fact that the road disturbance signal is amplified at these mentioned frequencies.

The same approach can be used to derive limitations for other transfer functions and various choices of the measured outputs.

It has been shown that the limitations always arise in the form of invariant frequency points (for example $\omega_1 = \sqrt{\frac{k_t}{m_u}}$ for $H^1_{zw}(j\omega_1)$, $\omega_2 = \sqrt{\frac{k_t}{m_u+m_s}}$ for $H^2_{zw}(j\omega_2)$ and $\omega_3 = 0$ for $H^3_{zw}(j\omega_3)$) and in the form of restricted rate of decay at frequencies tending to zero and infinity.

As an example, the complete sets of constraints for transfer functions $H^1_{zw}(s)$, $H^2_{zw}(s)$ and $H^3_{zw}(s)$ when suspension deflection and suspension deflection velocity are measured is as follows [2],[3]:

$H^1_{zw}(s)/_{s\to\infty} = O(s^{-3})$ (infinite freq. constraint)

$H^1_{zw}(s)/_{s\to 0} = 1+O(s^2)$ (zero freq. constraint)

$H^1_{zw}(j\omega_1) = -\frac{m_u}{m_s}$ for $\omega_1 = \sqrt{\frac{k_t}{m_u}}$

and analogically:

$$H^2_{zw}(s)/_{s\to\infty} = -\frac{k_t}{m_u}s^{-2} + O(s^{-3})$$

$$H^2_{zw}(s)/_{s\to 0} = O(s^2)$$

$H^2_{zw}(j\omega_2) = -(1+\frac{m_u}{m_s})$ for $\omega_2 = \sqrt{\frac{k_t}{m_u+m_s}}$ and

$$H^3_{zw}(s)/_{s\to\infty} = -1+\frac{k_t}{m_u}s^{-2} + O(s^{-3})$$

$$H^3_{zw}(s)/_{s\to 0} = -\frac{(m_u+m_s)}{k_t}s^2 + O(s^4)$$

$H^3_{zw}(j\omega_3) = 0$ for $\omega_3 = 0$.

It is often assumed that if practical difficulties are neglected, active systems could in principle produce arbitrary ideal behavior. This paper presents the factorization approach that is taken to derive limitations of achievable frequency responses for active vehicle suspension systems in terms of invariant frequency points and restricted rate of decay at high frequencies.

5 Bode Integral Theorem

To analyze the results given above it is useful to make a short review of some relevant ideas and definitions of sensitivity theory.

It is well known that the Bode sensitivity function can formally be extended to the characterization of the transfer function $F(s)$ with respect to the transfer function $G(s)$, called the variable component. Suppose that $G(s)$ is a transfer function of the controlled plant and $K(s)$ is a transfer function of the feedback controller. Then the Bode sensitivity function of the closed loop with the transfer function:

$$F(s) = \frac{G(s)}{1+G(s)K(s)} \quad (37)$$

is defined as:

$$S(s) = \frac{\partial \log F(s,G(s))}{\partial \log G(s)} = \frac{1}{1+G(s)K(s)} = \frac{1}{1+L(s)} \quad (38)$$

where $L(s) = G(s)K(s)$ is the open-loop transfer function. Because the open-loop (i.e. the control chain) sensitivity function $S_0(s)$ is equal to one in the whole frequency range, the sensitivity function $S(s)$ can serve as a criterion for the comparison of the sensitivity of the control loop with any control chain containing the same plant. For the frequencies ω, at which $|S(j\omega)| < 1$, the parameter sensitivity of the control loop is larger than that of the open control chain, i.e. the parameter sensitivity is increased by introduction feedback. To show this in terms of the Nyquist locus suppose the Nyquist locus of the open loop $L(j\omega)$ as shown in Fig. 3.

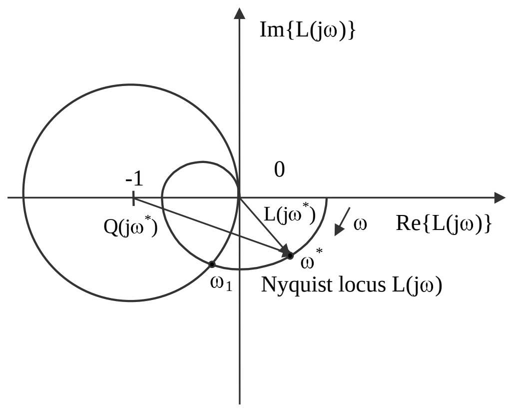

Fig. 3.: Evaluating the sensitivity from the Nyquist diagram

The vector from the point $(-1, j0)$ to the point ω^* represents the dominator of the closed-loop sensitivity function $S(j\omega^*)$.

Thus:

$$Q(j\omega^*) = 1 + L(j\omega^*) \quad (39)$$

This implies:

$$\left|S(j\omega^*)\right| = \left|Q(j\omega^*)\right|^{-1} \quad (40)$$

From Fig.3, we can therefore make the following observations. For those frequencies at which the locus $L(j\omega)$ cuts the unit circle centered at –1, the sensitivities of the closed-loop and open-loop systems are the same. For all those frequencies $\omega < \omega_1$ at which the locus $L(j\omega)$ lies outside the unit circle, $\left|S(j\omega)\right| < 1$ holds and the closed-loop system is less sensitive than the open-loop system. For all frequencies $\omega > \omega_1$, at which the locus $L(j\omega)$ lies inside the unit circle, $\left|S(j\omega)\right| > 1$ holds and the closed-loop system is more sensitive than the open one. From above the following can be seen: For a control system $L(j\omega)$ which has at least two more poles than zeros, there always exists a frequency ω_1 such that:

$$\left|S(j\omega)\right| > 1 \text{ for } \omega > \omega_1 \quad (41)$$

i.e. there exist frequencies at which the feedback control system is more sensitive than the uncontrolled one. The frequency range $\omega < \omega_1$ for which $\left|S(j\omega)\right| < 1$ holds can be influenced in a wide range by a proper choice of the transfer function $K(j\omega)$. This fact was already recognized by Bode and expressed by the following theorem:

Theorem 1. (Bode integral theorem): If the transfer function $L(s)$ of the open loop does not contain poles and $1 + L(s)$ does not contain zeros in the right half of the s-plane and if the number of poles of $L(s)$ exceeds the number of zeros at least by 2, then the following equality holds:

$$\int_0^\infty log\left|1 + L(j\omega)\right| d\omega = -\int_0^\infty log\left|S(j\omega)\right| d\omega = 0 \quad (42)$$

In words: For a stable control system with a pole excess of at least 2, the logarithm of the magnitude of the sensitivity function is on average equal to zero. This means that if the logarithmic Bode diagram $S(j\omega)$ is drawn, the area enclosed with 0 dB line for the region $\left|S(j\omega)\right| > 1$ is exactly the same as that for the region $\left|S(j\omega)\right| < 1$. The frequency ω_1 at which $\left|S(j\omega)\right| = 1$ can be specified by a suitably chosen transfer function $K(s)$ of the controller.

Bode original result was valid only for open-loop stable systems and was generalized by Freudenberg and Looze [1] for systems with unstable open loops as:

Theorem 2. (generalized Bode integral theorem): Assume that the open-loop transfer function $L(s)$ possesses finitely many open right half plane poles $\{p_i : i = 1,\dots,n \quad n \in N\}$, $Re(p_i) > 0$ including multiplicities. In addition, assume that:

$$\lim_{R \to 0} \sup_{|s| \geq R, Re(s) \geq 0} R\left|L(s)\right| = 0 \quad (43)$$

Then if the closed loop is stable, the sensitivity function must satisfy:

$$\int_0^\infty log\left|S(j\omega)\right| d\omega = \pi \sum_{i=1}^{n} Re(p_i) \quad (44)$$

The proof of this theorem is given by Freudenberg and Looze in [1].

6 Analysis of the Complete Sets of Limitations

In context with transfer functions $H(s)$ of the one-quarter-car model given in Section 1, the generalized Bode integral theorem can be modified as follows:

Theorem 3. Let RH_∞ is a set of rational transfer functions that are stable (their poles lie in the open right half-plane) and proper (the numerator degree of this functions is less than or equal to the denominator degree). Let $H(s)$ belongs to RH_∞ and satisfies $H(s)/_{s \to \infty} = -1 + O(s^{-2})$. Let $\{z_i, i = 1,..,n \quad n \in N\}$ are zeros of $H(s)$ with $Re(z_i) > 0$. Then:

$$\int_0^\infty log\left|H(j\omega)\right| d\omega = \pi \sum_{i=1}^{n} Re(z_i) \quad (45)$$

In control law design, it is desirable to prevent amplitudes of the dynamic responses $H^1_{zw}(s)$, $H^2_{zw}(s)$ and $H^3_{zw}(s)$ from being too large.

A brief examination of the results stated in Section 1 shows that the suspension deflection transfer function $H^2_{zw}(s)$ has modulus strictly grater than one for $\omega_2 = \sqrt{\frac{k_t}{m_u + m_s}}$ ($H^2_{zw}(j\omega_2) = -(1 + \frac{m_u}{m_s})$), i.e. at this frequency an amplification grater than one occurs. This amplification can be made less only and only by adjusting the ratio of the unsprung and sprung masses. From the result:

$$H^3_{zw}(s)/_{s\to\infty} = -1 + \frac{k_t}{m_u} s^{-2} + O(s^{-3}) \qquad (46)$$

it is evident, that $|H^3_{zw}(j\omega)|$ must tend to one from above as ω tends to ∞ and it turns out that $|H^3_{zw}(j\omega)|$ can not be made less than or equal to one at all frequencies. Since the right hand side of (45) is non-negative then it is not possible for $|H^3_{zw}(j\omega)|$ to be less then or equal to one at all frequencies since that would make the left hand side of the equation (45) negative. It has been shown in [3], that no matter what signals were used for feedback, the tyre deflection transfer function must amplify road disturbances at some frequencies. This fact is valid even for full state feedback used in the control loop.

A similar theorem is valid for transfer functions where $H(s)/_{s\to 0} = 1 + O(s^2)$

Theorem 4. Let $H(s)$ belongs to RH_∞ and satisfies $H(s)/_{s\to 0} = 1 + O(s^2)$.Let $\{z_i, i = 1,..,n \;\; n \in N\}$ are zeros of $H(s)$ with $Re(z_i) > 0$. Then:

$$\int_0^\infty log|H(j\omega)| \frac{d\omega}{\omega^2} = \pi \sum_{i=1}^{n} \frac{Re(z_i)}{|z_i|^2} \qquad (47)$$

Similarly to the consequences of theorem 3, the result $H^1_{zw}(s)/_{s\to 0} = 1 + O(s^2)$ from Section 1 is the case when $|H^1_{zw}(j\omega)|$ can not be less then or equal to one at all frequencies since that would make the left hand side of (47) negative. This fact is again valid even when full state feedback is introduced in the control loop.

In such cases that were mentioned above designers have the only possibility ”to shift” the frequencies where the amplifications occur to ”more advantageous places” or ”to spread” the ranges where amplifications occur making the amplification lower the positive area of the Bode magnitude plot, i.e. the area where $|H(j\omega)|$ is greater than one (is grater to at least the same extend than the negative area where $|H(j\omega)|$ is less than one) by choosing a proper feedback controller. Analogically, similar results can be derived for arbitrarily chosen measurements and load disturbances.

7 Results

Using a simple linear two degree-of-freedom car suspension system model (Fig.1) was shown that there still are limitations to suspension performance in the fully active state. It has been shown in the paper that there are some frequency points and frequency ranges where the transfer functions have modulus strictly greater than one i.e. where road and load disturbance amplification occur. On the base of the Bode integral theorems it has been shown that the transfer functions must be greater in modulus to at least the same extend that it is less than one, when measured in terms of the area on a Bode magnitude plot. In such a case there is a possibility to shift frequency ranges where disturbance amplification occurs to a ”more advantageous place” or to make magnitudes lower spreading the frequency range.

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